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ABSTRACT: A digital distributed parameter model for computing unsteady flow in liquid-filled fluid systems is presented. The wave-plan method employed in the model involves essentially the synthesis of incremental pressure pulses. The analysis is presented in a form general enough to be applied to a variety of hydraulic systems. To illustrate the application of the method to a specific system, the response of a long, straight liquid-filled line to sinusoidal inlet flow and pressure perturbations is computed. Both a constant-cross-section line and a tapered line are analyzed in the example.

KEY WORDS: hydraulics, unsteady flow, distributed parameter, nonlinear, digital.

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WAVE-PLAN ANALYSIS OF UNSTEADY FLOW

by Don J. Wood[1], A.M., A.S.C.E., Robert G. Dorsch[2], and Charlene Lightner[3]

SYNOPSIS

An analytical method for computing unsteady flow conditions in liquid-filled flow systems is developed. The method which is called the wave plan incorporates distributed parameter and nonlinear effects including the effects of viscous resistance. The wave plan is essentially a solution synthesized from the effects of incremental step pressure pulses. The pressure pulses are generated because of incremental flow-rate changes that originate in a hydraulic system from a variety of sources, including the mechanical motion of the system structure. The pressure pulses propagate throughout the system at sonic velocity and are partially transmitted and reflected at each discontinuity. The velocity change caused by each pressure pulse is obtained from the Joukowski relation. Pressure and velocity time histories at any point in the system are obtained by a timewise summation of the contributions of the incremental pressure pulses passing that point.

The analysis is presented in a form general enough to be applied to a variety of hydraulic systems. To illustrate the application of the method to a specific system, the response of a straight hydraulic line to a sinusoidal orifice-area variation of an upstream valve is computed. Both a constant-cross-section line and a tapered line are analyzed in the examples, and various nonlinear effects evaluated. Comparisons are carried out with experimental data obtained for the constant-diameter line and good agreement is shown to exist.

INTRODUCTION

Hydraulic systems employed in present day applications often require high performance with a small permissible range of variation from design flows and pressures. In addition, many hydraulic systems are structurally complicated. It is essential to be able to predict the dynamic response of these systems under transient and forced periodic conditions.

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The basic partial differential equations for unsteady flow are derived from continuity and momentum relations. Closed-form distributed parameter solutions for small sinusoidal flow perturbations in long lines having negligible fluid damping are available and are in good agreement with experimental data[4-6]. In general, however, the nonlinear partial differential equations cannot be solved without resorting to tedious numerical or graphic techniques.

An analytical study was therefore undertaken at the NASA Lewis Research Center to develop a numerical distributed-parameter solution for unsteady flow in hydraulic systems that would be in a form readily lending itself to digital-computer computation. The analysis techniques which were developed will be referred to herein as the wave-plan. The wave-plan method has some similarities to a method of characteristics solution due to the technique of tracing sonic disturbances throughout the system; however, the wave-plan method is more easily applied to complex unsteady flow systems in addition to lending itself better to physical interpretation.

This paper presents the details of the wave-plan analysis along with the basic elements of the corresponding digital-computer program required to calculate unsteady flow in liquid systems. The analysis is presented in a form general enough to be applied to a variety of fluid systems. Examples are given to illustrate the application of the method to specific hydraulic systems.

ANALYSIS

General

Basic equations. - The pressure and velocity of the liquid within a line as a function of position and time can be obtained from the equations of momentum and continuity.

The momentum equation for a one-dimensional elastic fluid of constant mean density is[7]

$$\frac{\partial H}{\partial x} = -\frac{1}{g}\left[\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + f(v)\right] \tag{1}$$

where $f(v)$ is the resistance due to fluid viscosity, which is some function of velocity. (Symbols are defined in appendix D.)

The continuity equation for an elastic liquid in an elastic line is[7]

$$\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} = -\frac{C^2}{g}\frac{\partial V}{\partial x} \tag{2}$$

Equations (1) and (2) are nonlinear partial differential equations which, when solved along with the boundary conditions in a liquid-filled line, give relations for velocity and pressure in the line.

Method of solution. - The direct solution of the continuity and momentum equations to obtain velocities and pressures is difficult even when the boundary conditions are relatively simple (open or closed ends). In addition, exact solutions have been found only for a system where the resistance term is linearized or neglected entirely[4-6].

Acceptable solutions of nonidealized unsteady flow problems including nonlinear effects require the use of numerical methods and a digital computer. The continuity and momentum equations have been solved by a numerical technique based on the method of characteristics and adapted for digital computer use[7]. Identical solutions may be obtained more readily by an alternate process. This method, referred to as the wave-plan solution, is developed in this paper.

A wave-plan solution is obtained as follows: At the point or points in a liquid system where a disturbance is introduced (such as an oscillating valve or a moving impedance change), an incremental change in liquid flow rate due to the disturbance over a very short interval of time is computed. The incremental pressure pulse accompanying the flow-rate change is then computed. This pressure pulse is propagated throughout the system at sonic speed. The pressure pulse is partially reflected and partially transmitted at all geometrical and physical discontinuities in the fluid network. Pressure and velocity time histories are computed for any point in the system by summing with time the contributions of incremental waves.

Characteristic impedance. - The relation (characteristic impedance) between pressure and velocity changes caused by a pulse traveling in the liquid-filled line is computed from momentum considerations. Figure 1 shows pressure and flow conditions, as a pressure wave propagates in a liquid-filled line, that exist at time t and at time $t + \Delta t$. The wave takes the time Δt to travel the distance Δx from A to B. During this time there is a pressure $P + \Delta P$ on the left end and a pressure P on the right end of the liquid contained between points A and B. This unbalanced pressure causes the fluid to accelerate. Newton's second law gives

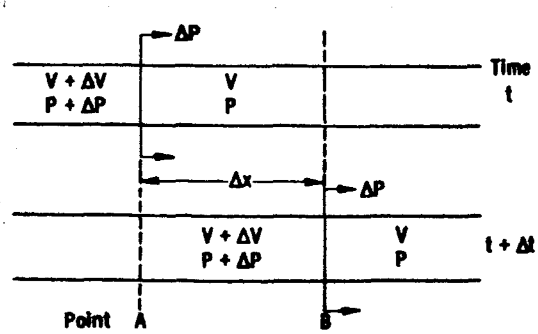


Figure 1. - Effect of pressure pulse on mean line conditions.

$$(P + \Delta P - P)A = \rho A\ \Delta x \frac{\Delta V}{\Delta t}$$

Canceling and rearranging yield

$$\Delta P = \rho \, \Delta V \frac{\Delta x}{\Delta t}$$

The term $\Delta x/\Delta t$ is the propagation speed of the pressure wave. The wave speed is equal to the sonic velocity C in the system if the mean velocity of the liquid in the line is neglected. Since the mean velocity of the liquid is usually much smaller than the acoustic velocity, this is usually permissible. Thus,

$$\Delta P = \rho C \, \Delta V \tag{3a}$$

or in terms of head of liquid

$$\Delta H = \frac{C}{g} \Delta V \tag{3b}$$

The sonic speed C for a liquid flowing within a line is influenced by the elasticity of the line wall, and for a system that is axially unrestrained[7-9] can be calculated from

$$C = \sqrt{\frac{E_f}{\rho} \Bigg/ \left(1 + \frac{E_f D}{E_c \tau}\right)} \tag{4}$$

Equation (3), which is the well known Joukowski equation, states that the pressure change at any point in a line is equal to the product of the velocity change at the point times the characteristic acoustic impedance ρC of the liquid in the line. This relation may also be derived by employing continuity and energy principles[8, 9].

Generation of Pressure Waves

Pressure waves may originate in liquid flow systems in a variety of ways. Two common sources are an orifice or valve with a varying open area and a moving point in a line where there is an impedance change. In order to apply the wave-plan method to varying area change or moving impedance change, the corresponding disturbance function is approximated by a series of discrete changes over small equal time intervals. Figure 2 shows this approximation.

Each of these changes is assumed to occur instantaneously at some time during the time interval. The pressure perturbations associated with each of these changes

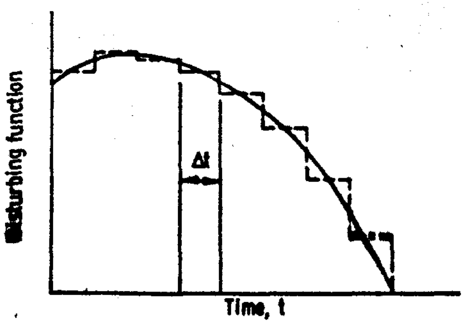


Figure 2. - Step approximation of disturbing function.

are then computed as shown in the following examples.

Varying area orifice. - A pressure wave is generated at an orifice that experiences a change in opening area. The magnitude of the generated wave is a function of conditions before the area change and the magnitude of the area change.

The flow through an orifice is assumed to obey the square-law relation:

$$V = B\sqrt{\Delta H}$$

The orifice coefficient B is a function of the open area, the line area, and the discharge coefficient and is easily determined for a particular orifice. For the case of an open area that is a small percentage of the line area ($A_0 << A$), the orifice coefficient is given closely by

$$B = C_D \frac{A_0}{A} \sqrt{2g}$$

where the discharge coefficient C_D is a function of Reynolds Number (and hence, velocity). For small velocity perturbations, however, the discharge coefficient varies only slightly, and hence the orifice coefficient may be considered to vary linearly with area changes. The examples in this report deal with this case.

Figure 3 shows conditions at an orifice before and after a small orifice-area change. The flow is from left to right. The external pressure head may be varied in a prescribed manner.

The momentum equation across the wave front gives

$$\Delta H_{11} = \frac{c}{g}(V_1 - V_{11})$$

The flow out the orifice after the orifice area change is given by

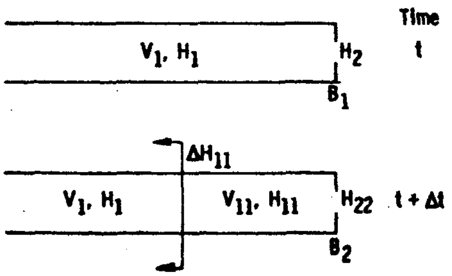


Figure 3. - Effect of step change in orifice coefficient at terminal orifice.

$$V_{11} = B_2\sqrt{H_1 + \Delta H_{11} - H_{22}}$$

Solving the momentum and orifice equations simultaneously gives the following quadratic in V_{11}

$$V_{11}^2 + bV_{11} + c = 0 \qquad (5)$$

where

$$b = \frac{B_2^2 C}{g}$$

$$c = -B_2^2 \left(H_1 + \frac{CV_1}{g} - H_{22} \right)$$

The positive root of this equation gives the desired value for outflow velocity. Substituting this value for V_{11} back into the momentum equation gives the magnitude of the pressure wave generated by a sudden change in orifice coefficient from B_1 to B_2.

Moving terminal orifice. - The analysis of the magnitude of a pressure wave generated by the motion of a component in a fluid line is exemplified by considering the motion of a terminal orifice. The lateral velocity of the orifice is so represented by a series of step changes that the velocity of the orifice remains constant over short time intervals. Figure 4 shows conditions at a terminal orifice just before and a short time after a step change in velocity. The orifice coefficient need not be constant.

The momentum equation across the generated wave is

$$\Delta H_{11} = \frac{C}{g}(V_1 - V_{11})$$

The displacement shown in figure 4 takes place in the time increment Δt. The continuity equation states that the net inflow across boundary AA equals the net flow out the orifice plus the storage that takes place during the time interval. Therefore

$$V_{11}A\ \Delta t = VE_2 A\ \Delta t + VOA\ \Delta t$$

or

$$V_{11} = VE_2 + VO$$

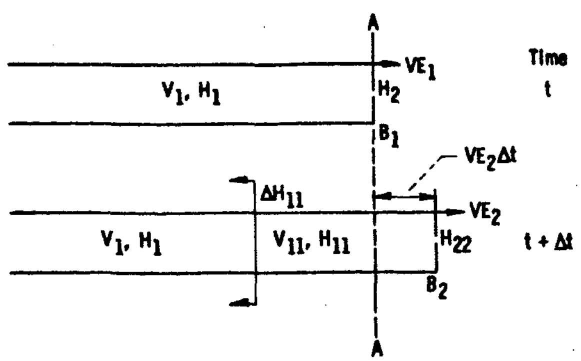


Figure 4. - Effect of lateral motion of terminal orifice.

where VE_2 is the lateral velocity of the orifice end during the time interval, and VO is the liquid velocity in the line (with respect to the orifice).

The flow out the orifice is given by

$$VO = B_2 \sqrt{H_1 + \Delta H_{11} - H_{22}}$$

Solving the preceding equations simultaneously for the resulting line velocity gives the following quadratic in VO:

$$VO^2 + bVO + c = 0 \tag{6}$$

where

$$b = \frac{B_2^2 C}{g}$$

and

$$c = -B_2^2\left[H_1 + \frac{C}{g}(V_1 - VE_2) - H_{22}\right]$$

Effect of Viscous Resistance

After the generation of a pressure wave at a perturbing point in a liquid system, the wave propagates with sonic velocity throughout the system. The viscous resistance of the liquid medium influences the propagation of this wave.

The effect of viscous resistance on pressure pulse propagation can be neglected without appreciable effect for short, large-diameter lines where such losses are small; however, for longer lines or small-diameter lines carrying viscous liquid, the resistive losses are not negligible and should be included in an unsteady flow analysis.

The exact solution of the partial differential equations (1) and (2) is limited by the necessity of linearizing or neglecting the friction term entirely. For graphical analyses it has been necessary to consider the friction losses as lumped at one or more points in the pipeline if a manageable solution is to be obtained[10].

To include the effects of viscous resistance in a wave-plan solution, the extent that fluid viscosity influences the propagation of a pressure pulse must be determined. There is presently no experimental or analytical information available which is in a form that can be used for the prediction of the effect of viscous resistance in unsteady line flow. Because of this limitation it is necessary to make a quasi-steady approximation and to assume that viscous losses in a small line length dx are given at any instant by the Darcy equation as

$$\Delta h_L = \frac{f\, \Delta x\, V^2}{2gD} \tag{7}$$

The form of equation (7) is identical to that of an internal square-law orifice if the orifice coefficient is given by

$$B_F = \sqrt{\frac{2gD}{f\ \Delta x}}$$

This equation implies that viscous losses over a small length of line can be lumped at an internal square-law orifice with a properly chosen orifice coefficient. This representation is referred to as the "orifice analogy," and its use in graphical analysis has been suggested by Bergeron[10].

Line losses can be distributed at many discrete points along a line by inserting a large number of correctly chosen square-law (friction) orifices. Impinging waves are then reflected and transmitted at these friction orifices in a manner similar to that of reflection and transmission through a small region of flowing viscous liquid. The equations governing wave reflection and transmission at a friction orifice are developed in the next section.

Reflection and Transmission of Pressure Waves at System Discontinuities

It is necessary at each discontinuity to compute the magnitude of transmission and reflection of each pressure pulse in terms of initial conditions at a discontinuity, the nature of the discontinuity, and the magnitude of the impinging waves. These derivations are all made by employing the three fundamental fluid flow relations (continuity, momentum, and energy).

Figure 5 shows two waves impinging and reflecting simultaneously at a discontinuity in a liquid system. The notation used in this figure is employed in subsequent specific examples. Liquid moving from left to right has a positive magnitude of velocity.

The momentum equations across the wave fronts yield

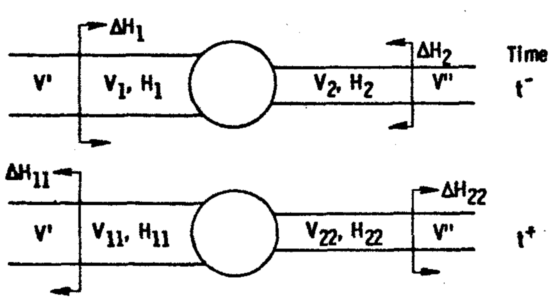


Figure 5. - Nomenclature for conditions before and after wave action at discontinuity.

$$\Delta H_1 = \frac{C_1}{g}(V' - V_1) \tag{8}$$

$$\Delta H_{11} = \frac{C_1}{g}(V' - V_{11}) \tag{9}$$

$$\Delta H_2 = \frac{C_2}{g}(V_2 - V'') \tag{10}$$

$$\Delta H_{22} = \frac{C_2}{g}(V_{22} - V'') \tag{11}$$

(See fig. 5 for definition of V' and V''.) Equations (8) and (9) combine to give

$$\Delta H_{11} = \Delta H_1 + \frac{C_1}{g}(V_1 - V_{11}) \tag{12}$$

and equations (10) and (11) give

$$\Delta H_{22} = \Delta H_2 + \frac{C_2}{g}(V_{22} - V_2) \tag{13}$$

The new pressures are given by

$$H_{11} = H_1 + \Delta H_1 + \Delta H_{11} \tag{14}$$

$$H_{22} = H_2 + \Delta H_2 + \Delta H_{22} \tag{15}$$

Terminal orifice. - A terminal orifice is considered to be bounded by a pressure reservoir. The pressure in the reservoir may be changing in a prescribed manner. In addition, the orifice coefficient may also be changing in a prescribed manner. Figure 6 shows conditions before and after reflection from an orifice bounded at the inlet by a pressure reservoir.

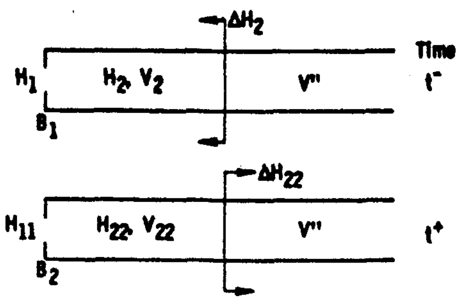


Figure 6. - Pressure-pulse reflection at terminal orifice.

If the discharge after wave action is in the positive direction (from the reservoir to the line), the head-discharge relation for the orifice is given by

$$V_{22} = B_2 \sqrt{H_{11} - (H_2 + \Delta H_2 + \Delta H_{22})}$$

Solving this equation simultaneously with the momentum equations (13) for the velocity gives the following quadratic in V_{22}:

$$V_{22}^2 + bV_{22} + c = 0 \tag{16}$$

where

$$b = \frac{B_2^2 C_2}{g}$$

and

$$c = -B_2^2\left(H_{11} - H_2 - 2\,\Delta H_2 + \frac{C_2 V_2}{g}\right)$$

Because the resulting direction of flow was taken as positive, the positive root of equation (16) is the desired result.

The magnitude of the reflected pressure wave is obtained from equation (13). The new pressure at the orifice is given by equation (15).

If the direction of flow is into the reservoir after wave action (negative direction), the velocity head relation is given by

$$V_{22} = B_2 \sqrt{(H_2 + \Delta H_2 + \Delta H_{22}) - H_{11}}$$

Simultaneous solution with the momentum equations gives a quadratic in V_{22}:

$$V_{22}^2 + bV_{22} + c = 0 \tag{17}$$

where

$$b = -\frac{B_2^2 C_2}{g}$$

$$c = B_2^2 \left(H_{11} - H_2 - 2\,\Delta H_2 + \frac{C_2 V_2}{g} \right)$$

The negative root of equation (17) is the desired result.

It is necessary to determine the resulting direction of flow so that the correct equation (eq. (16) or (17)) can be applied. Inspection of equation (16) discloses that the term $H_{11} - H_2 - 2\,\Delta H_2 + C_2 V_2/g$ must be positive to yield the necessary positive root. If this term is negative, the flow must be into the reservoir, equation (17) must be used, and the negative root must be computed.

Friction orifice. - Figure 7 shows conditions before and after pulse-wave action at a friction orifice. Because of identical conduits on both sides of the orifice, the head-velocity relation for the orifice after wave action for flow from left to right is

$$V_{11} = V_{22} = B_F \sqrt{H_1 + \Delta H_1 + \Delta H_{11} - H_2 - \Delta H_2 - \Delta H_{22}}$$

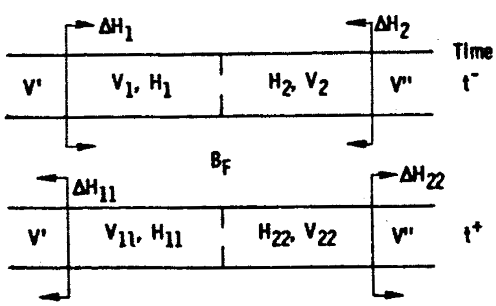


Figure 7. - Pressure-pulse reflection at internal orifice.

Noting that $C_1 = C_2 = C$ and solving this equation simultaneously with the momentum equations (12) and (13) give

$$V_{11}^2 + bV_{11} + c = 0 \tag{18}$$

where

$$b = \frac{2B_F^2 C}{g}$$

$$c = -B_F^2\left(H_1 + 2\,\Delta H_1 - H_2 - 2\,\Delta H_2 + \frac{2CV_1}{g}\right)$$

The positive root of this quadratic equation gives the resulting velocity through the friction orifice. The term $H_1 + 2\,\Delta H_1 - H_2 - 2\,\Delta H_2 + 2CV_1/g$ must be positive to yield a positive root. If this term is negative, the resulting flow is from right to left, and the head-velocity relation for the orifice is

$$V_{11} = V_{22} = B_F\sqrt{H_2 + \Delta H_2 + \Delta H_{22} - H_1 - \Delta H_1 - \Delta H_{11}}$$

This equation combines with the momentum equations to give the following:

$$V_{11}^2 + bV_{11} + c = 0 \qquad (19)$$

where the coefficients b and c are of the same magnitude and of opposite sign from the corresponding coefficients in equation (18).

The negative root of this equation gives the resulting velocity in the negative direction. The magnitudes of the resulting pressure waves and the pressures after wave action are given by equations (12) to (15).

Diameter discontinuity. - The relation between net head and discharge for a diameter change in a conduit is derived by utilizing energy relations. Figure 8 shows the notation for this derivation.

If $A_1 > A_2$, the diameter discontinuity is an abrupt constriction, and the energy equation for flow through the constriction from left to right after wave action is

$$H_{11} + \frac{V_{11}^2}{2g} = H_{22} + \frac{V_{22}^2}{2g} + 0.5\left(1 - \frac{A_2}{A_1}\right)^2 \frac{V_{22}^2}{2g}$$

The area ratio is denoted as

$$R = \frac{A_1}{A_2} \qquad (20)$$

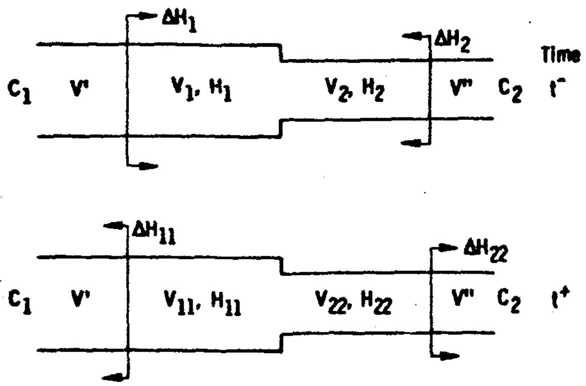


Figure 8. - Pressure-pulse reflection at diameter discontinuity.

Then the continuity equation is

$$V_{22} = RV_{11} \tag{21}$$

If these relations are substituted into the energy equation and the terms are combined, the following net head-velocity relation is obtained:

$$H_{11} - H_{22} = V_{11}^2 \left(\frac{3R^2 - 2R - 1}{4g} \right) \tag{22}$$

If $A_1 < A_2$, the diameter discontinuity is an abrupt expansion. The energy equation for flow through the expansion from left to right after wave action is

$$\frac{V_{11}^2}{2g} + H_{11} = \frac{V_{22}^2}{2g} + H_{22} + \left(\frac{A_2}{A_1} - 1 \right)^2 \frac{V_{22}^2}{2g}$$

This equation gives the following velocity-head relation for an expansion:

$$H_{11} - H_{22} = V_{11}^2 \left(\frac{R^2 - R}{g} \right) \tag{23}$$

The case of a lossless diameter discontinuity is also important because step diameter changes can be employed to simulate a tapered line. In the continuously tapered line the losses are small and sometimes can be neglected. The energy equation without the loss term is

$$\frac{V_{11}^2}{2g} + H_{11} = \frac{V_{22}^2}{2g} + H_{22}$$

The resulting velocity-head relation is

$$H_{11} - H_{22} = V_{11}^2 \left(\frac{R^2 - 1}{2g} \right) \tag{24}$$

The final net head after wave action can be written as

$$H_{11} - H_{22} = H_1 + \Delta H_1 + \Delta H_{11} - H_2 - \Delta H_2 - \Delta H_{22}$$

or

$$H_{11} - H_{22} = H_1 + 2\,\Delta H_1 + \frac{C_1}{g}(V_1 - V_{11}) - H_2 - 2\,\Delta H_2 - \frac{C_2}{g}(V_{22} - V_2) \tag{25}$$

The relation is combined with the continuity equation and the velocity-head relation to obtain second-order polynomials for the line velocity after wave action. The equation is

$$aV_{11} + bV_{11} + c = 0 \tag{26}$$

where

$$b = \frac{C_1}{g} + \frac{C_2 R}{g}$$

$$c = -\left(H_1 + 2\,\Delta H_1 + \frac{C_1 V_1}{g} + \frac{C_2 V_2}{g} - H_2 - 2\,\Delta H_2\right)$$

The coefficient a depends on the type of discontinuity and whether energy losses are included. For a contraction with losses included the coefficient is

$$a = \frac{3R^2 - 2R - 1}{4g} \tag{27}$$

For an expansion with losses included the coefficient is

$$a = \frac{R^2 - R}{g} \tag{28}$$

For either type of discontinuity with no losses included the coefficient is

$$a = \frac{R^2 - 1}{2g} \tag{29}$$

For each case the resulting velocity is the positive root of equation (26). Therefore, the term

$$H_1 + 2\,\Delta H_1 + \frac{C_1 V_1}{g} + \frac{C_2 V_2}{g} - H_2 - 2\,\Delta H_2$$

must be positive. If this term is negative, the resulting flow is from right to left, and the equations are reformulated by putting the loss term on the correct side of the energy equation.

Junctions of three or more legs. - The previously presented mathematical analyses pertaining to the reflection of a pressure pulse at a diameter discontinuity have taken into account energy losses introduced by the discontinuity. With junctions of three or more legs, the inclusion of energy loss terms becomes increasingly difficult because of the irreversibility of frictional effects. Thus, the treatment employed herein will be similar to the one presented in standard references on fluid dynamics and will not be derived[11].

The relations for computing the percentages of magnitude of a pressure wave transmitted and reflected at a junction are based on the following:

(1) The Joukowski equation applied across each wave

(2) Continuity of flow at the junction

(3) Continuity of pressure at the junction

The magnitude of the wave that is transmitted to all the other legs of a junction of n legs due to a wave of magnitude ΔH impinging in line i is given by $T(i)\Delta H$, where the transmission coefficient $T(i)$ is given by

$$T(i) = \frac{\dfrac{2A(i)}{C(i)}}{\displaystyle\sum_{j=1}^{n} \frac{A(j)}{C(j)}} \tag{30}$$

The magnitude of the wave reflected in leg i is given by $R(i)\Delta H$ when the reflection coefficient $R(i)$ is given by

$$R(i) = T(i) - 1$$

Analysis of the Dynamics of a Liquid System by Wave Plan

The wave-plan analysis supposes a system composed of a discrete number of discontinuities connected by lossless line segments, which serve only to transmit pressure pulses. The discontinuities include geometric ones, such as terminal orifices and diameter discontinuities, and artificial ones, such as friction orifices. A typical simulated fluid system is shown in figure 9.

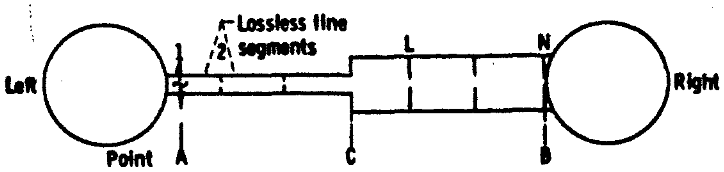


Figure 9. - Wave-plan representation of typical fluid system.

There is a variable-area orifice at A (valve) and a fixed orifice at B. At C there is an abrupt change in line diameter. Friction orifices are inserted between A and C and between C and B to simulate viscous resistance.

The equations developed thus far have applied only to conditions at a particular point in a system at a particular instant of time. Through the use of these equations it is possible to compute velocity, pressures, and magnitudes of pressure pulses leaving a particular discontinuity in terms of the magnitudes of impinging waves and conditions prior to wave action.

To analyze a system, however, the equations must be both time and position dependent. This dependency is indicated by the form of the basic partial differential equations (eqs. (1) and (2)). It is apparent that the pressure wave impinging at one discontinuity in a liquid system is exactly that wave which left an adjacent discontinuity at some time in the past (because of the lossless line connector simulation). Thus, the waves are related by position and time. In general, variables have the form $f(x,t)$. Specifically, $H(x,t)$, $V(x,t)$, and $\Delta H(x,t)$ denote the pressure head, velocity, and pressure pulse as a function of position and time, with x and t being the position and time subscripts, respectively.

The wave-plan analysis provides for information at specified points in the system (discontinuities) and at discrete time intervals. New information is available only at discrete time intervals because all of the disturbing functions are approximated by a series of small step changes occurring at specified times. The system response to these disturbing functions will also be in the form of step changes. Because of the two-parameter dependence of the dependent variables, it is advantageous to denote position and time subscripts.

Position subscripts. - In order to apply the analysis equations that have been developed for a discrete point to a liquid system, it is necessary to introduce a method of denoting position. This is done by numbering adjacent discontinuities in a consecutive manner as indicated in figure 9. (This arrangement is modified at junctions of three or more legs). The position subscript is denoted as L.

In addition, because pressure and velocity states may differ across discontinuities, it is necessary to denote a left and a right side of a discontinuity. This is done by adding an L for left and an R for right to the nonsubscripted part of the variable.

Thus $HL(L,t)$ denotes the pressure head to the left of discontinuity L at time t.

Time subscripts. - The time subscript J is defined as follows:

$$t = J\,\Delta t$$

where t is the time and Δt is the working time increment. Thus HR(L, J) denotes the pressure head to the right of discontinuity L at time $t = J\,\Delta t$.

Selection of working time increment. - The working time increment Δt is the time interval between successive computations.

Its selection is determined by two factors. First, the time increment must be small enough to approximate accurately all disturbing functions by a series of step changes. Second, it is necessary that all reflections of pressure pulses in the system take place at an integer number of time intervals. The wave travel times between all adjacent discontinuities will then be an integer number of working time increments. If this were not the case, waves would be impinging on a discontinuity in a completely arbitrary fashion and would make the solution unmanageable.

The selection of the time increment is simplified because the majority of discontinuities in a system will be friction orifices, and these can be placed where desired. For example, the selection of the working time interval for the system shown in figure 9 is made as follows:

(1) The wave travel times between adjacent geometric discontinuities are determined:

(a) tac - wave travel time between A and C

(b) tcb - wave travel time between C and B

Within the desired limits, the largest time interval of which these travel times are integer multiples is determined. This time interval represents the largest working time increment possible.

(2) It is then determined if the integer number of working time increments necessary for travel between A and C and between C and B is large enough to insert the desired number of friction orifices that must, of course, be separated in time by at least one working time increment. If not, the increment can be divided by any integer to obtain a smaller working time increment.

(3) It must be determined if the working time interval necessary to assure an integer number of time intervals between all discontinuities is small enough to approximate accurately the disturbing function (variable-area orifice at A) by a series of step changes. If not, the working time increment is further divided by an integer necessary to get a suitable approximation. The integer number of time increments between discontinuities is increased by this factor.

Subscripted notation for analysis equations. - The analysis equations are easily extended for application to a liquid system by the use of the subscripted notation. Figure 10 shows conditions at a discontinuity before and after wave action when the subscripted notation is employed.

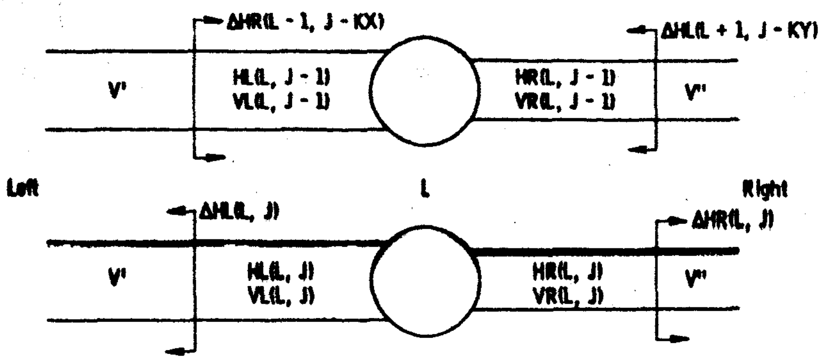


Figure 10. - Subscripted nomenclature for conditions before and after wave action at discontinuity.

The time that the waves reach discontinuity L is $t = J\,\Delta t$. Conditions before and after that time are constant, and step changes occur at $t = J\,\Delta t$.

A comparison of the nonsubscripted notation in figure 5 to the subscripted notation in figure 10 gives the identities necessary to make the analysis equations general expressions for a liquid system. These identities are as follows:

VL(L, J - 1) = V_1 = velocity to left of discontinuity L

HL(L, J - 1) = H_1 = pressure head to left of discontinuity L

VR(L, J - 1) = V_2 = velocity to right of discontinuity L

HR(L, J - 1) = H_2 = pressure head to right of discontinuity L

ΔHR(L - 1, J - KX) = ΔH_1 = pressure pulse coming from adjacent discontinuity to left. KX is the number of working time intervals it takes a sonic disturbance to travel between the two discontinuities.

ΔHL(L + 1, J - KY) = ΔH_2 = pressure pulse coming from adjacent discontinuity to right. KY is the number of working time intervals it takes a sonic disturbance to travel between these two discontinuities.

The conditions after wave action are as follows:

VL(L, J) = V_{11} = velocity to left of discontinuity L

HL(L, J) = H_{11} = pressure head to left of discontinuity L

VR(L, J) = V_{22} = velocity to right of discontinuity L

HR(L, J) = H_{22} = pressure head to right of discontinuity L

ΔHL(L, J) = ΔH_{11} = pressure pulse leaving discontinuity L and moving to left

ΔHR(L, J) = ΔH_{22} = pressure pulse leaving discontinuity L and moving to right

Substitution of these identities into the analysis equations (eqs. (8) to (29)) yields a perfectly general set of algebraic equations applicable to any liquid system.

Method of computation. - The solution is carried out by using the subscripted equations as follows:

(1) A working time increment is selected.

(2) Initial conditions are computed from given steady-state values that exist in the

system prior to the initiation of a disturbance. These conditions include velocity and pressure to the left and right of each discontinuity and are denoted at $t = 0$ by VL(L,O), VR(L,O), HL(L,O), and HR(L,O).

(3) Computations are then carried out to determine conditions at each discontinuity at the end of the first working time interval by using the conditions from step (2) as initial conditions. It should be noted that the analysis equations will predict no change at an undisturbed discontinuity that is not subjected to impinging waves. Thus, conditions will only change at the end of the first time interval at a disturbance source (point A, fig. 9). At the disturbance source conditions change, and waves are generated that will subsequently alter conditions at adjacent discontinuities.

(4) Each discontinuity is analyzed at the end of each interval. The computations are carried out as long as desired to give pressures and velocities at each discontinuity at the end of each working time interval.

Digital Computer Programing

The wave-plan analysis consists of the sequential solution of many equations. The dynamic analysis of even a rather simple liquid system for a reasonable number of time intervals involves a large number of computations. A digital computer must be employed if the solution is to be obtained in a reasonable length of time.

To make the digital analysis as general as possible, computer segment programs (subroutines) have been written for the most common types of discontinuities. These discontinuity solutions can then be combined (incorporating junction equations, if necessary) to obtain digital computer models for various liquid flow systems.

Each discontinuity subroutine computes conditions at the discontinuity for some point in time as a function of the local velocity and pressure-head conditions a time interval earlier, the magnitude of impinging waves (from adjacent discontinuities), and the step changes in disturbing functions during the time interval.

The following subroutines have been written to be used in the digital model representation:

(1) Terminal orifice subroutine

(2) Internal (friction) orifice subroutine

(3) Diameter discontinuity subroutine

Each subroutine uses the general relations given by equations (12) to (15). Also each subroutine initially assigns nonsubscripted identities to their subscripted counterparts and finally assigns the nonsubscripted results to the subscripted terms (as indicated in the section entitled Subscripted notation for analysis equations). A description and computer printouts of the digital computer subroutines are included in appendix A. All digital computer programming has been done in Fortran IV.

The subroutines are combined to form digital computer models for the dynamic analysis of liquid flow systems. Two examples are included in the next section.

RESULTS AND DISCUSSION

The equations and programs developed in the ANALYSIS section form a basis for the analysis of unsteady flow conditions in a liquid flow network. The equations and programs have been kept general so that they may be applied to the analysis of a variety of liquid systems. In general, the wave-plan digital-computer analysis has distinct advantages over other methods such as analog-computer models that substantially linearize or lump parameters and small perturbation techniques that linearize around mean-line conditions. The following features of the wave-plan analysis should be noted:

(1) Viscous friction effects are easily included.

(2) Perturbing functions of any form may be used.

(3) Nonlinear relations are easily included. (Linearization of parameters is of little or no advantage.)

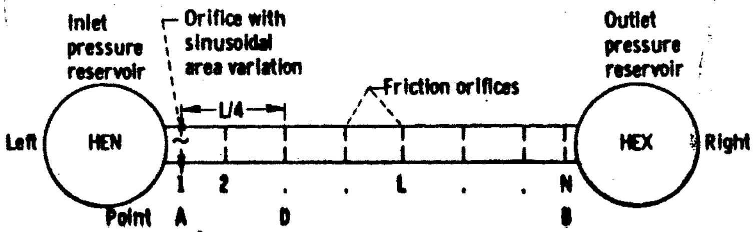
(4) Both transient and steady-state responses to disturbing functions are given.

(5) Complex networks can be analyzed.

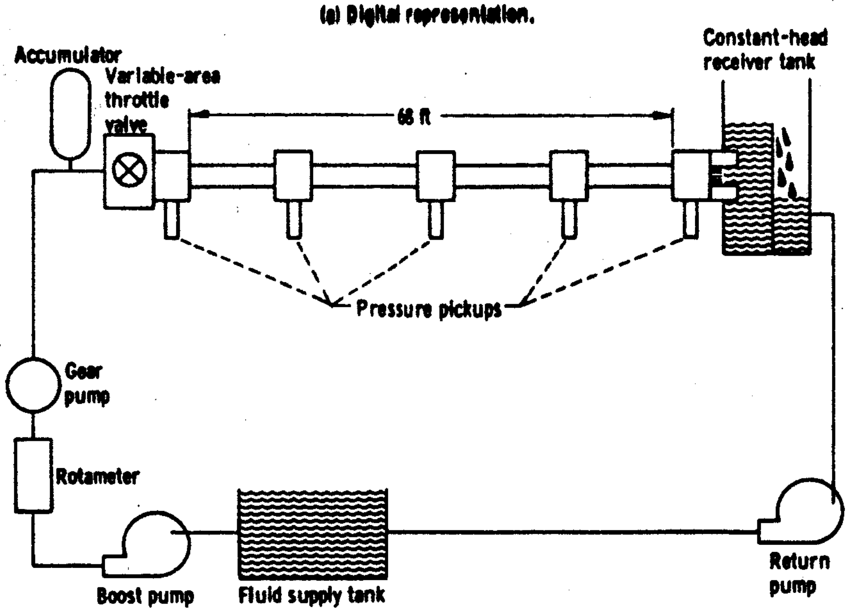
Two examples will be used to illustrate the application of the wave-plan method to the construction of digital models of particular liquid flow systems. The first example will illustrate the method of computing the transient and steady-state response of a long, straight hydraulic line to a periodic sinusoidal flow disturbance. In the second example the response of a long tapered line will be computed.

Example I: Long Hydraulic Line With Oscillating Inlet Orifice

This example was chosen because the computer results can be compared directly with experimental data from the Lewis Research Center line dynamics rig[4-6]. Schematics of the liquid flow system are given in figure 11. Figures 11(a) and (b) show the analytical model and the experimental rig, respectively. The system consists of a 68-foot-long, 7/8-inch-inside-diameter line with pressure reservoirs at both ends. The test fluid was JP-4 fuel or another hydrocarbon, depending on the liquid viscosity desired. Line pressures ranged from 200 to 400 pounds per square inch. A variable-area throttle valve that perturbs the system is located at the upstream end of the line (point A), and a fixed orifice is located at the downstream end (point B). The details of constructing the digital model are given in appendix B along with the program constants used in the calculations.



(a) Digital representation.



(b) Schematic of experimental rig.

Figure 11. - Hydraulic line with oscillating inlet orifice.

Effect of amplitude of input perturbations. - Most line-dynamics analyses are based on small-perturbation techniques that permit the use of linearized terminal impedances as an approximation for nonlinear pressure-flow relations at the end of the line. Therefore, in taking experimental line-dynamics data[4-6] the practice is to limit the amplitude of the oscillating throttle disturbance generator. The minimum usable amplitude, however, for a given line condition and terminal impedance is limited in practice by the ratio of the sine wave signal to the nonharmonic noise which decreases with decreasing amplitude. The permissible maximum amplitude is limited in practice by the appearance of harmonics in the sinusoidal pressure signal as the amplitude is increased. The system must be operated between these two limits.

The relation between the disturbance amplitude and the degree of nonlinearity of the pressure and velocity perturbations was determined analytically by varying the amplitude of the sinusoidal input perturbations for the analytical line model. The mean orifice coefficients were 0.8 at the inlet and 1.0 at the outlet. The steady line velocity was 14 feet per second. For the digital computer program (appendix B) the amplitude of the input orifice coefficient perturbation BA was varied at 40 cps as shown in table I.

TABLE I. - AMPLITUDES OF ORIFICE COEFFICIENT PERTURBATIONS

Case	Input orifice coefficient perturbation amplitude, BA	Percent of steady-state orifice coefficient
1	0.48	60
2	.32	40
3	.20	25
4	.15	18.75

The calculated steady-state responses for the pressure and fluid velocity at the inlet and outlet of the line (points A and B, fig. 11(a)) are shown in figure 12 (p. 22). The velocities and pressures are dimensionless, having been divided by steady line values. Figure 12 shows that, as the amplitude of the disturbing function is decreased, the system responses are more nearly described by a sinusoidal-type periodic function; however, for the particular line condition and terminal impedance used in the example, nonlinearities are evident even for relatively small perturbations. For example, the inlet velocity perturbation is very clearly nonsinusoidal for an 18.75-percent disturbing amplitude even though the perturbation amplitude is only 2 percent in the positive direction and 4 percent in the negative direction.

Transient response. - Dynamic systems analyses are usually based on steady-state responses to steady sinusoidal inputs. This analysis technique does not, however, predict the transient response to the initiation of a disturbance or a change in the disturbance characteristics in a liquid flow system. The wave-plan solution gives both transient and steady-state solutions. This point is clarified by examining both the transient and final steady-state responses to the initiation of a sinusoidal disturbing function in a liquid system under steady mean-flow conditions. A value for BA of 0.15 (case 4, table I) was chosen for the amplitude of the disturbance function. The line response starting with the initiation of the disturbing function is given in figure 13 (pp. 23 and 24). For the case studied several cycles were required for the perturbation velocities and pressures to approach the steady-state values. In some cases the amplitudes of the transient responses are considerably different from the steady responses. For example, during the early part of the transient period the magnitude of the inlet velocity perturbation was nearly double the steady-state value.

Comparison of analytical and experimental results. - The experimental hydraulic line (fig. 11(b)) was operated with the upstream throttle valve sinusoidally perturbed at 70 cps with area perturbation amplitudes of 12.3, 33.8, 55.3, and 76.8 percent of the open area. The mean value of the coefficient for the upstream orifice was 0.65, and the corresponding value for the downstream orifice was 0.42. The downstream orifice was slightly open with respect to characteristic. Figure 14 (p. 25) shows the resulting experimental pressure-time oscilloscope traces for the quarter-length point of the line (measured from the upstream throttle valve).

The experimental conditions were duplicated by the digital model, and figure 15 (p. 25) shows the theoretical results for these four different input amplitudes. The

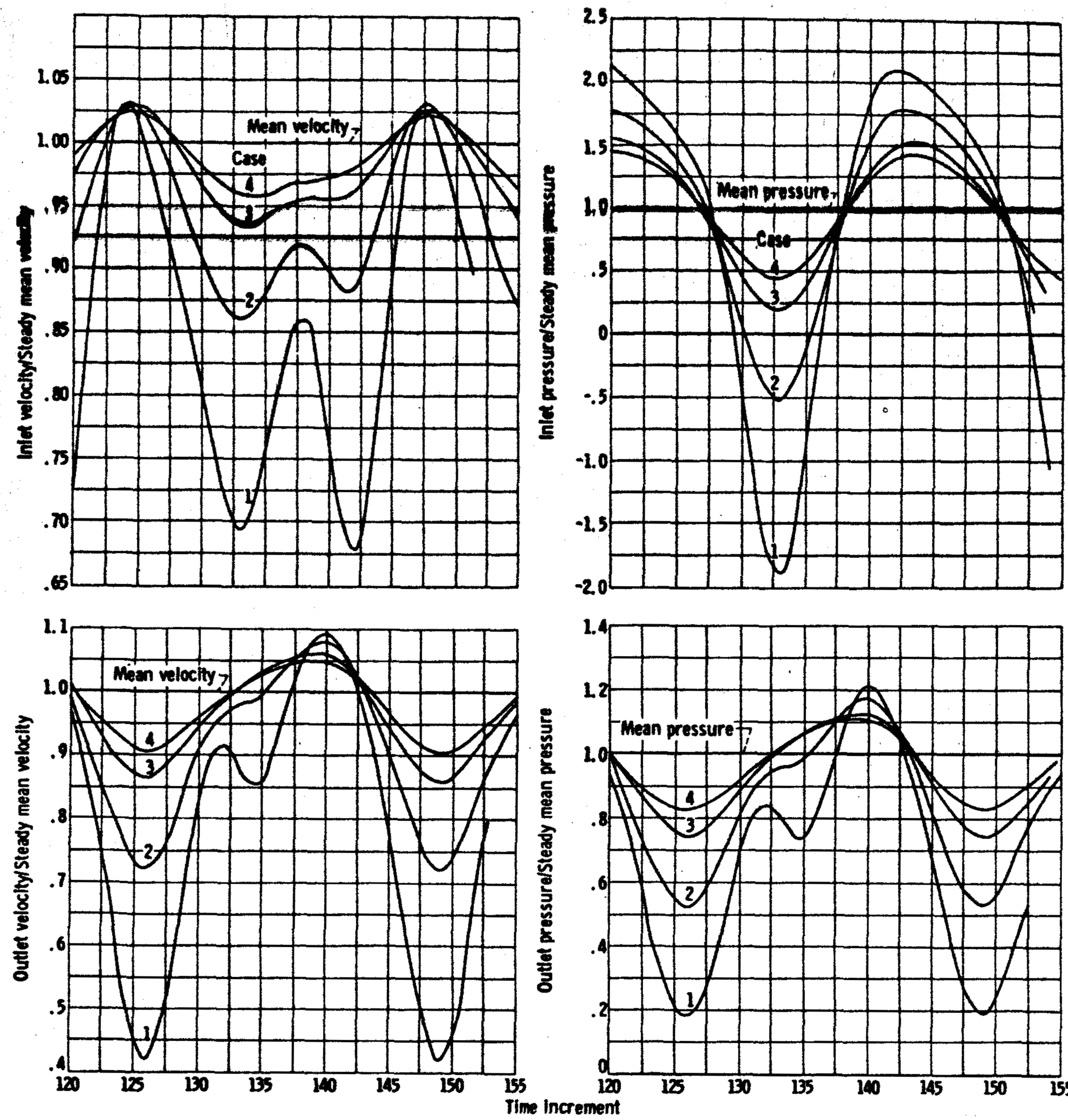


Figure 12. - Effect of amplitude of inlet orifice area perturbation on steady-state response of long line.

agreement between the wave shapes determined analytically and experimentally is very good for all cases. In addition the analytical and experimental magnitudes agree to within the accuracy of the measurements (approx. 3 percent).

This example illustrates that the wave-plan analysis for unsteady liquid flow is capable of accurately including nonlinear effects and giving results in good agreement with experimentally observed values.

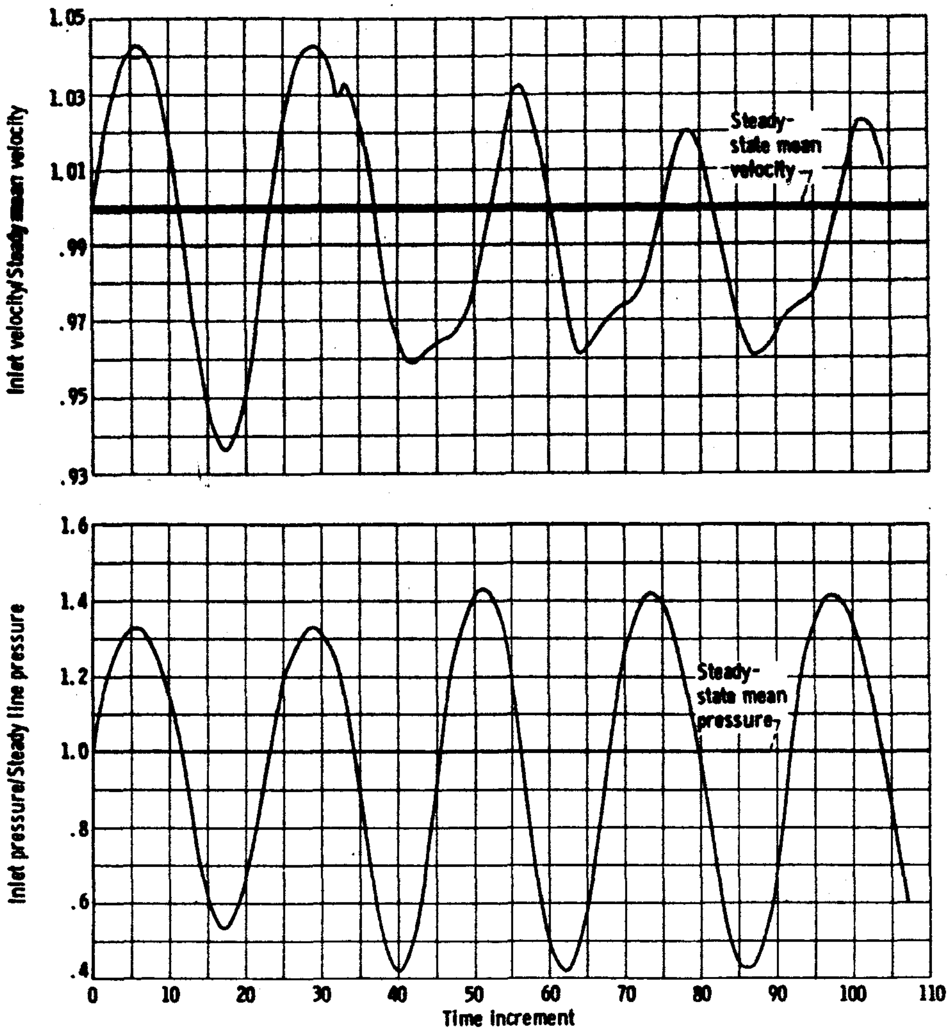


Figure 13. - Transient response of long line to initiation of sinusoidal inlet valve area perturbation.

Example 2: Response of Long Tapered Line to Sinusoidal Input Disturbance

The analysis of the response of a liquid system composed of lines of nonuniform diameter (tapered) to a periodic disturbing function is difficult if classical methods are employed; however, tapered lines can easily be included in a wave-plan analysis by approximating the tapered line by a line composed of a discrete number of diameter changes (fig. 16, p. 26).

In order to evaluate the quantitative effect of tapered lines, the system shown in figure 17 (p. 26) was studied in detail. This system consists of two pressure reservoirs connected by a tapered line. At the end of the tapered lines, relatively large resistive losses were introduced in the form of square-law orifices. The throttle valve at point A has a sinusoidally varying area, and the orifice at point B is fixed.

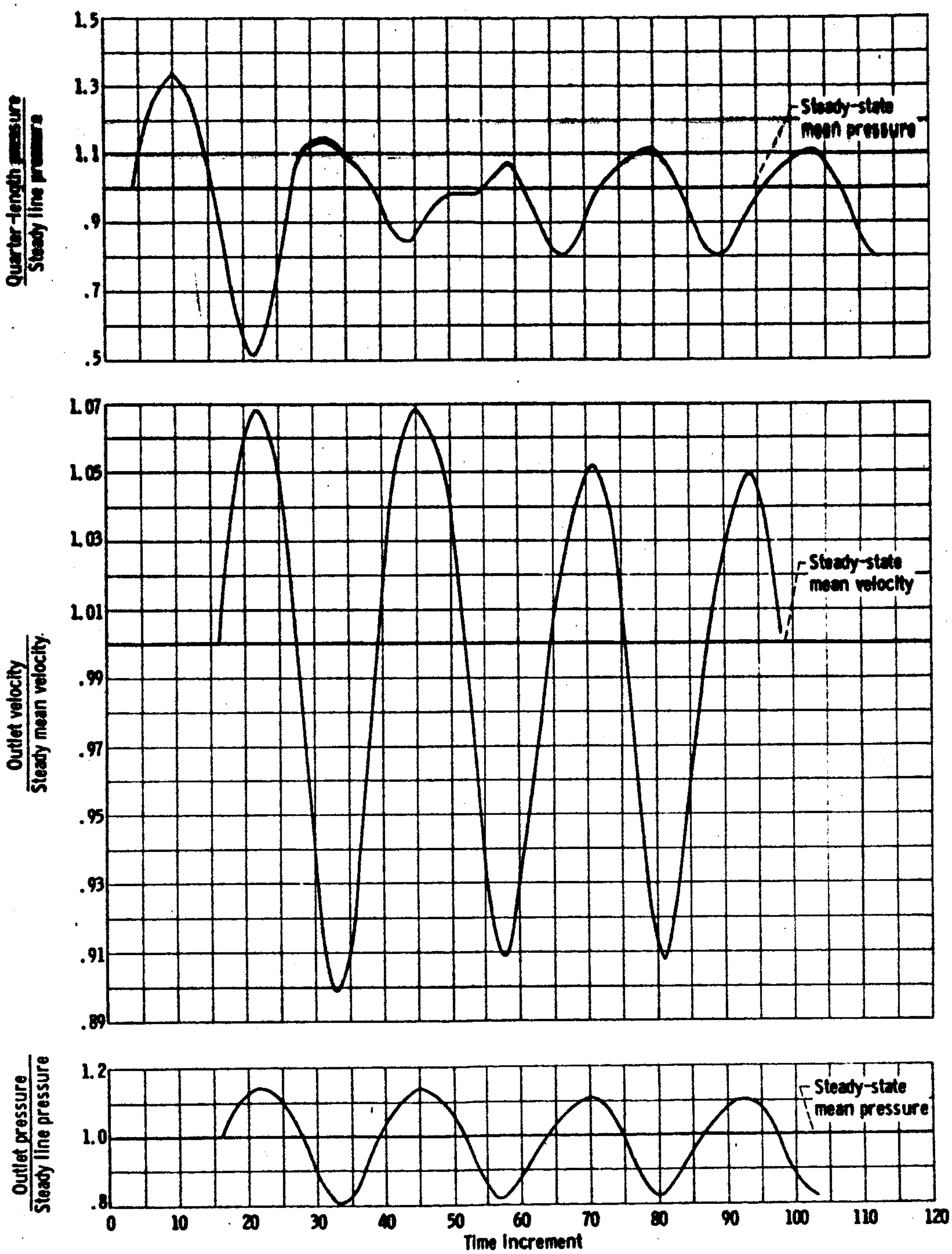
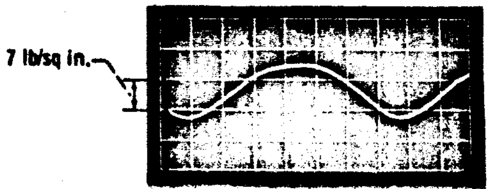
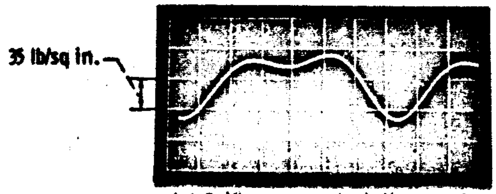


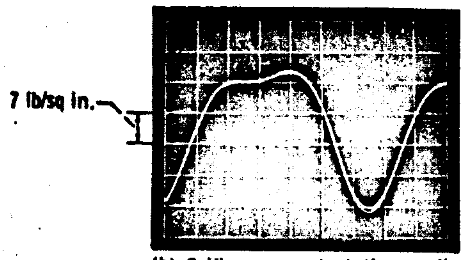
Figure 13. - Concluded. Transient response of long line to initiation of sinusoidal inlet valve area perturbation.



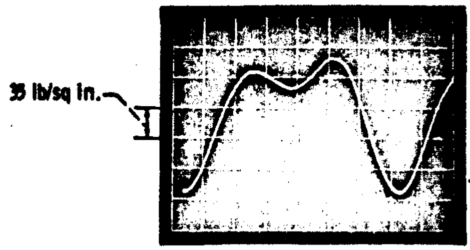
(a) Orifice area perturbation amplitude, 12.3 percent of open area; maximum amplitude (peak to peak) of pressure perturbation, 11.2 ± 0.7 pounds per square inch.



(c) Orifice area perturbation amplitude, 55.3 percent of open area; maximum amplitude (peak to peak) of pressure perturbation, 73.5 ± 3.5 pounds per square inch.

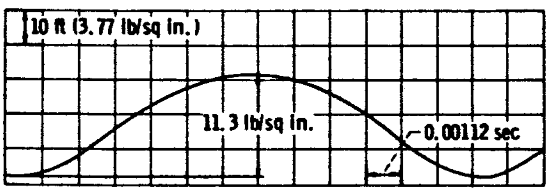


(b) Orifice area perturbation amplitude, 33.8 percent of open area; maximum amplitude (peak to peak) of pressure perturbation, 32.2 ± 0.7 pounds per square inch.

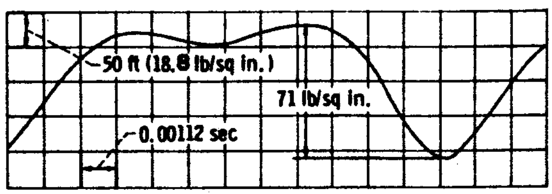


(d) Orifice area perturbation amplitude, 76.8 percent of open area; maximum amplitude (peak to peak) of pressure perturbation, 147 ± 3.5 pounds per square inch.

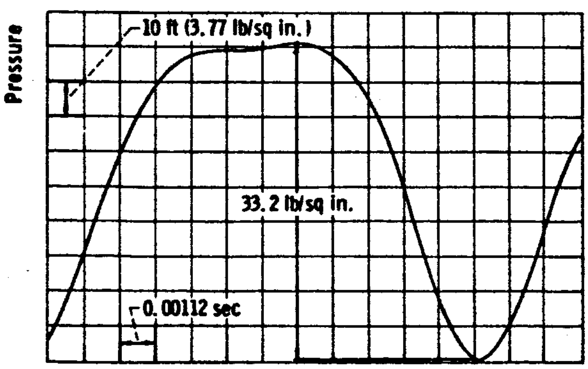
Figure 14. - Experimental pressure-time oscilloscope traces for quarter-length point of line.

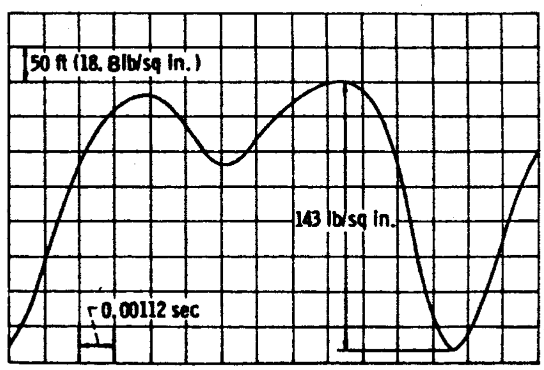


(a) Orifice area perturbation amplitude, 12.3 percent of open area.



(c) Orifice area perturbation amplitude, 55.3 percent of open area.





Time

(b) Orifice area perturbation amplitude, 33.8 percent of open area.

(d) Orifice area perturbation amplitude, 76.8 percent of open area.

Figure 15. - Analytical pressure-time traces for quarter-length point of line.

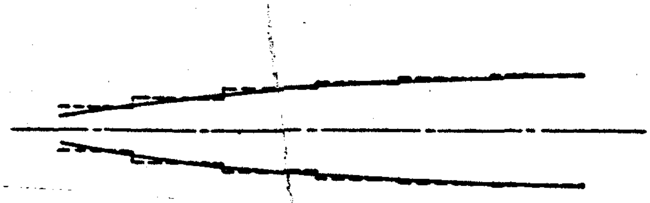


Figure 16. - Step-diameter-change approximation to tapered line.

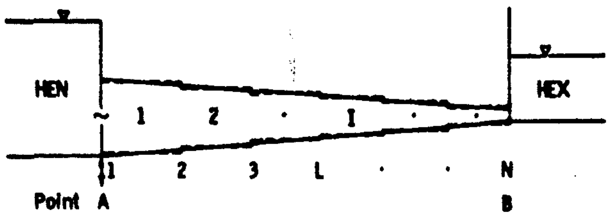


Figure 17. - Reservoir conduit system with tapered line.

This system was studied for various degrees of taper. The configurations were chosen to give the diameter at the inlet as

$$DA = \bar{D} - \delta \quad \text{ft} \qquad (31a)$$

and the diameter at the outlet as

$$DB = \bar{D} + \delta \quad \text{ft} \qquad (31b)$$

The diameter of the tapered conduit is assumed to vary linearly between the ends, and the average diameter is always $\bar{D}$ feet.

If the resistances of the terminal orifices are large compared to the line loss, the steady discharge of the system is practically independent of δ and would depend mainly on the reservoir pressures and the mean line diameter. Even for lines with much smaller end-resistive losses the system discharge would be only slightly dependent on the degree of taper, because line losses consisting of friction and expansion or contraction losses would be mainly dependent on line length and average diameter. Mathematically this can be written as

$$Q^2 = \frac{HEN - HEX}{K_{en} + K_l + K_{ex}} \qquad (32)$$

where K_{en}, K_l, and K_{ex} are the loss coefficients of the entrance orifice, the line, and the exit orifice, respectively.

When this system is used with large values of K_{en} and K_{ex} as compared to K_l, it is possible to compare directly the dynamic response of systems that are essentially statically equivalent. Pressure and flow perturbations can be quantitively compared as a function of degree of taper. The straight conduit ($\delta = 0$) is a special case that is included in the analyses.

The tapered conduit was chosen so that the wave velocity would be constant; thus the ratio of diameter to wall thickness remained constant along the length of the line.

For the digital model the tapered line was approximated by N - 1 short straight sections of conduit, as shown in figure 17. In this manner the diameter of each approximating section I is computed by the equation

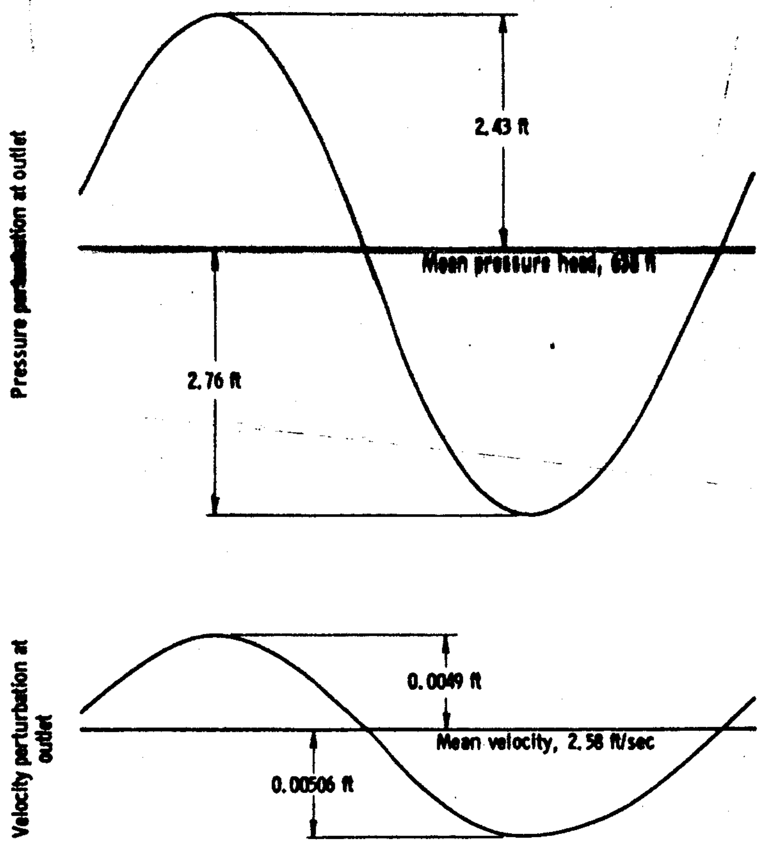
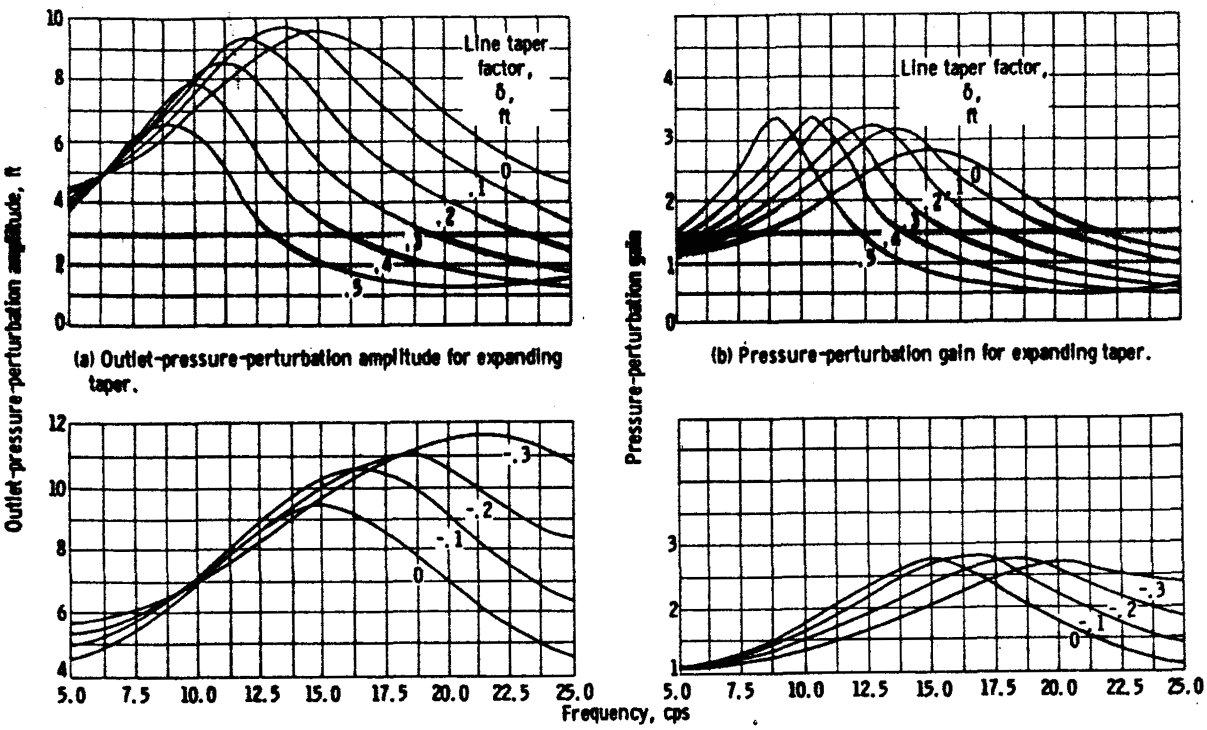


Figure 18. - Pressure and velocity perturbation responses in tapered line. Inlet diameter, 0.6 foot; exit diameter, 1.4 feet; frequency, 17.5 cps.

$$D(I) = DA + \left(\frac{DB - DA}{2}\right)\left(\frac{2I - 1}{N - 1}\right) \tag{33}$$

The details of the digital model along with the computer flow diagram and program constants are given in appendix C. One basic system was studied in detail, a system having a high-resistance orifice (three-fourths to seven-eighths closed geometrically but slightly open with respect to characteristic impedance) at the inlet with a 10-percent sinusoidal variation in the area of the orifice opening. The outlet orifice was chosen of even higher resistance than the inlet and represents an end that is closed with respect to characteristic impedance. Nine degrees of taper were studied, the average diameter $\overline{D}$ being 1 foot for each case. The values of δ that were used were -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, and 0.5 foot.

The quarter-wave resonance frequency for the nontapered line is 15 cps. Frequencies of 5, 7.5, 10, 12.5, 15, 17.5, 20, and 25 cps were studied for each case. In all cases the pressure and flow perturbations were slightly nonlinear even though they were small. In all cases they were within 2 percent of mean-line values. Figure 18 shows typical pressure and flow perturbations for one cycle.



(a) Outlet-pressure-perturbation amplitude for expanding taper.

(b) Pressure-perturbation gain for expanding taper.

(c) Outlet-pressure-perturbation amplitude for contracting taper.

(d) Pressure-perturbation gain for contracting taper.

Figure 19. - Frequency dependence of tapered-line dynamic response.

In spite of the nonlinearity of the perturbations, they were steady and repeatable after several cycles (during which the transient died out) and resembled sinusoids. It was considered appropriate to make comparisons based on positive amplitudes. This procedure is followed in the ensuing discussion.

The analytical results are displayed graphically in figure 19. Figure 19(a) shows the variation of the amplitude of the outlet pressure perturbation with frequency for the expanding taper (line that expands from inlet to outlet). Figure 19(b) shows the pressure perturbation gain (ratio of outlet pressure perturbation amplitude to inlet pressure perturbation amplitude) for the expanding taper. Figures 19(c) and (d) show the same results for contracting tapers. The most striking result apparent in figure 19 is the shift in the resonant frequency for tapered lines. This shift is shown in figure 20.

Another result is that the maximum pressure perturbations at the outlet have a strong dependence on δ and vary considerably over the entire frequency range. Thus a certain amount of control can be exerted over the response of this system by the judicious choice of tapers. For example, in the case of an expanding taper of $\delta = 0.3$ the amplitude of the outlet pressure perturbation is significantly reduced when compared to a straight-line response for the frequencies shown except in the range from 6 to 12 cps. A line with a contracting taper of $\delta = -0.3$ differs little from a nontapered line ($\delta = 0$) up to 15 cps. For frequencies greater than 15 cps the outlet pressure amplitude pertur-

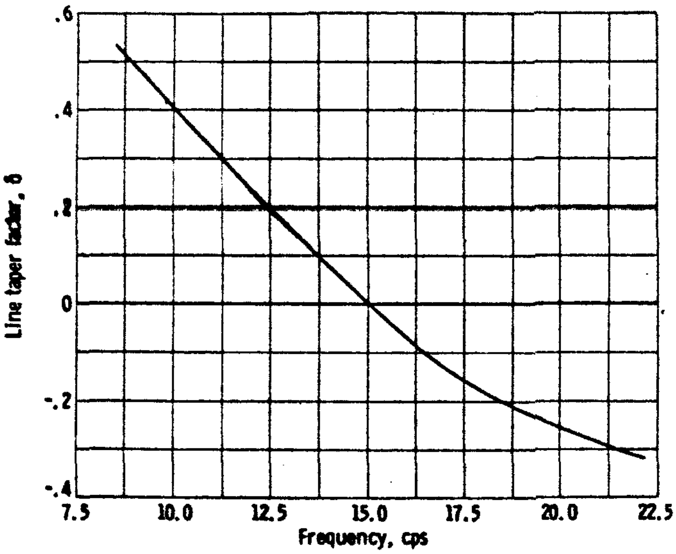


Figure 20. - Variation of resonant frequency with taper factor.

bation is considerably greater.

This example indicates that the dynamic response of a liquid system can be altered significantly with a certain amount of control without significantly altering the static response of the system; however, the primary intent of the example is to illustrate the use of the wave-plan analysis. Although the example was applied to a system with a linear taper and a sinusoidal disturbing function, a system of any arbitrary taper with any disturbing function could have been analyzed with no additional difficulty.

CONCLUSIONS

The analytical method presented in this paper provides a means of obtaining distributed parameter solutions to a variety of unsteady flow problems for liquids flowing in conduits. An advantage of the method is that the complete solution is obtained. For example, both the transient and the steady-state response to a suddenly imposed periodic flow disturbance is obtained. Furthermore, the disturbing function can be of arbitrary form and need not be periodic. Nonlinear effects are easily included. In addition, the wave-plan method is advantageous in making certain types of dynamic response calculations. For example, the response of liquid-filled lines having types of axial cross-sectional area distributions for which there would be little hope of obtaining closed-form analytical solutions can be easily handled.

The wave-plan analysis can readily solve problems in which there is interaction between the structural motion of the conduits and the perturbations in the fluid flowing within the conduits. Interactions of this type often occur in hydraulic systems; however, the motion is usually neglected in the analyses.

APPENDIX A

COMPUTER SUBROUTINES

Digital-computer routines have been written to solve for conditions at various line discontinuities after wave action in terms of the conditions at the discontinuity prior to wave action, the magnitudes of impinging waves, and the physical characteristics of the discontinuity. These routines have been formulated in such a manner that they may be easily incorporated into a computer program to synthesize different liquid flow systems.

In all cases the computer routines have been written in nonsubscripted notation, which corresponds exactly to the notation presented in the text of this paper. In the calling vector that calls the subroutine into the main program, the subscripted counterparts of the nonsubscripted subroutine variables are identified.

This mechanism of identifying subscripted variables in a calling vector makes it possible to use the analyses presented in the text for specific orientations and flow conditions to solve for all orientations and flow conditions. In this manner the analyses presented in the text of this paper have been generalized. Proper identification of the variables in the calling vector is tantamount to orienting the discontinuity to correspond to the case analyzed in the text. A table incorporated into each subroutine gives the proper subscripted-to-nonsubscripted dependence for each case.

Each of the analyses presented in the text resulted in a polynomial that is quadratic in a velocity term. For several reasons these equations were solved by an iterative manner by employing Newtonian extrapolation. The roots of the polynomials were determined by looping the following equations:

$$\text{error} = \frac{aV11^2 + bV11 + c}{2aV11 + b}$$

$$V11 = V11 - \text{error}$$

The looping is continued until the error is of sufficiently small magnitude. An acceptable solution can be obtained within a very few cycles because the wave-plan solution deals with small changes, and the value of the velocity before wave action is a good approximation of the final value. In addition, the coefficients b and c are generally large and always much larger than a. These conditions lead to a rapidly diminishing error term.

A direct solution of the quadratic equations by the quadratic formula results in a solution that is a small difference between two large numbers. In some cases sufficient accuracy cannot be obtained even when double-precision computing techniques are employed. The computer programs which use notation identical to that appearing in the text of this paper didactically illustrate the logic of the subroutines and are presented in lieu of flow diagrams. A brief description of the program and the Fortran IV program for each computer routine follows.

Terminal Orifice Subroutine

Equations (16) and (17) are the basic equations employed in this analysis. They are derived in the text for an orifice on the left end of a line. The table at the beginning of the subroutine gives the identities that must be substituted into the calling vector depending on whether the orifice is on the right or left end of the line. Before the subroutine is entered, the following terms must be identified with a numerical value:

B	orifice coefficient
HEN	reservoir head for left end
HEX	reservoir head for right end
J	time counter
K	number of time increments to nearest discontinuity
L	position counter

The computer program for a terminal orifice is as follows:

```
      SUBROUTINE TOR(H2,DH2,V2,H11,H22,DH22,V22)
C      SUBROUTINE TERMINAL ORFICE
      COMMON/BOX/A,B,C, G,BF,ERROR
C                LEFT END                RIGHT END
C      H2          HR(L,J-1)            HL(L,J-1)
C      V2          V(L,J-1)            -V(L,J-1)
C      DH2         DHL(L+1,J-K)         DHR(L-1,J-K)
C      V22         V(L,J)              -V(L,J)
C      H22         HR(L,J)              HL(L,J)
C      DH22        DHR(L,J)             DHL(L,J)
C      H11         HEN                   HEX
      BB = C*B*B/G
      CC=-B*B*(H11-H2-2.*DH2+C*V2/G)
      IF(CC) 10,10,4
    4 BB = -BB
      CC = -CC
   10 V22= V2
   11 ER = (V22*V22+BB*V22+CC)/(2.*V22+BB)
      V22= V22-ER
      IF (ABS(ER)-ERROR ) 12,12,11
   12 DH22=DH2+C/G*(V22-V2)
      H22=H2+DH2+DH22
      RETURN
      END
```

Friction Orifice Subroutine

The basic equations for this routine are equations (18) and (19). Before this routine is entered, the following terms must have a numerical value:

BF friction orifice coefficient

J time counter

K number of time increments to nearest discontinuity

L position counter

The computer routine is the following:

```
      SUBROUTINE FOR(H1,DH1,DH2,H2,V1,H11,DH22,DH11,H22,V11)
C     SUBROUTINE FRICTION ORFICE
C      H1        HL(L,J-1)
C      DH1       DHR(L-1,J-K)
C      DH2       DHL(L+1,J-K)
C      H2        HR(L,J-1)
C      V1        V(L,J-1)
C      H11       HL(L,J)
C      DH11      DHL(L,J)
C      DH22      DHR(L,J)
C      H22       HR(L,J)
C      V11       V(L,J)
      COMMON/BOX3/ A, B, C,G,BF,ERROR
      BB = 2.*BF*BF*C/G
      CC=-(H1+2.*DH1+2.*C/G*V1-(H2+2.*DH2))*BF*BF
      IF(CC) 3,3,2
    2 BB = -BB
      CC = -CC
    3 V11 = V1
    4 ER = (V11*V11+BB*V11+CC)/(BB+2.*V11)
      V11 = V11-ER
      IF (ABS(ER)-ERROR ) 5,5,4
    5 DH11 = DH1+C/G*(V1-V11)
      H11 = H1+DH1+DH11
      DH22 = DH2+C/G*(V11-V1)
      H22 = H2+DH2+DH22
      RETURN
      END
```

Diameter Discontinuity Subroutine

Equations (20) to (29) are employed in the diameter discontinuity subroutine. These equations are derived for flow in the positive direction (to the right) for both an expansion and a contraction. Before the subroutine is entered, the direction of flow after wave action is determined, and this direction in turn determines which subscripted variables are inserted into the calling vector.

The subroutine also includes the case where no viscous loss is considered across the discontinuity. To utilize this case, it is necessary to assign the term LOSS a value of zero. Additional terms that must be defined are the following:

J time counter

K number of time increments to nearest discontinuity

L position counter

The computer routine is as follows:

```
      SUBROUTINE DISC(H1,V1,DH1,H2,V2,DH2,C1,C2,A1,A2,H11,DH11,H22,DH22,
     1V11,V22)
C  DISCONTINUITY  SUBROUTINE
      COMMON/BOX/A,B,C, G,BF,ERROR
      COMMON/BOX2/ LOSS
C                FLOW TO RIGHT               FLOW TO LEFT
C               CCC GREATER THAN 0          CCC LESS THAN 0
C     H1         HL(A,J-1)                   HR(L,J-1)
C     V1         VL(L,J-1)                   -VR(L,J-1)
C     DH1        DHR(L-1,J-K)                DHL(L+1,J-K)
C     H2         HR(L,J-1)                   HL(L,J-1)
C     V2         VR(L,J-1)                   VL(L,J-1)
C     DH2        DHL(L+1,J-K)                DHR(L-1,J-K)
C     H11        HL(L,J)                     HR(L,J)
C     V11        VL(L,J)                     -VR(L,J)
C     DH11       DHL(L,J)                    DHR(L,J)
C     H22        HR(L,J)                     HL(L,J)
C     V22        VR(L,J)                     -VL(L,J)
C     DH22       DHR(L,J)                    DHL(L,J)
C     C1         CL(L)                       CR(L)
C     C2         CR(L)                       CL(L)
C     A1         AL(L)                       AR(L)
C     A2         AR(L)                       AL(L)
      R=A1/A2
      CC= -(H1+2.*DH1+C1*V1/G-H2-2.*DH2+C2*V2/G)
      BB = C1/G+C2*R/G
      IF (LOSS) 4,3,4
    3 AA = (R*R-1.)/(2.*G)
      GO TO 10
    4 IF(R-1.) 1,1,2
    1 AA = (R*R-R)/G
      GO TO 10
    2 AA = (3.*R*R-2.*R-1.)/(4.*G)
   10 V11= V1
    5 ER = (AA*V11*V11+BB*V11+CC)/(2.*AA*V11+BB)
      V11=V11-ER
      IF (ABS(ER)-ERROR) 6,6,5
    6 DH11 = DH1+C1/G*(V1-V11)
      V22=R*V11
      DH22=DH2+C2/G*(V22-V2)
      H11=H1+DH1+DH11
      H22=H2+DH2+DH22
      RETURN
      END
```

APPENDIX B

DIGITAL MODEL OF HYDRAULIC LINE WITH OSCILLATING INPUT ORIFICE

A schematic drawing of the system chosen for this example is shown in figure 11 (p. 20).

The system consists of a variable-area orifice at A that perturbs the system and a fixed orifice at B. The conduit is bounded by pressure reservoirs. (The reservoir pressures can vary.)

To keep the solution general, N - 2 friction orifices are inserted, which result in N discontinuities. A time interval is so chosen that each discontinuity is one time interval from adjacent discontinuities. Thus

$$\Delta t' = \frac{L}{C}\left(\frac{1}{N - 1}\right)$$

This time increment is a multiple of the working time increment Δt, which is chosen small enough so that all disturbing functions can be accurately described by step functions.

Initial values for velocity through each discontinuity and the pressure to the right and left of each discontinuity must be inserted into the computer as data or the computation of these quantities from the initial steady condition must be integrated into the computer program.

The disturbing function can either be read into the computer as a function of time or it can be computed. For this case a sinusoidal orifice coefficient for the inlet (at A) is taken to be

$$B = BO + BA\ SIN(2\pi\ FJ\Delta t)$$

The orifice coefficient for the outlet is considered to be constant and is denoted by BN.

The following data are needed as input to the computer:

C	wave velocity in conduit, ft/sec
L	length of line, ft
BO	mean orifice coefficient when subjected to sinusoidal perturbations
BO1	steady-state inlet orifice coefficient
BA	amplitude of inlet orifice coefficient perturbation
F	frequency of pressure perturbation

BN outlet orifice coefficient
VO steady line velocity, ft/sec
HEN pressure head of inlet reservoir, ft
HEX pressure head of outlet reservoir, ft
g acceleration due to gravity, ft/sec^2
N number of discontinuities
NN number of working time increments for wave to travel length of conduit
K number of working time increments between discontinuities
M total number of time increments for which computations will be made
N2 number of friction orifices for which pressure and velocity data will be printed

In setting up the computer program the notation $j = x, y, z$ is used. The notation means that computations are carried out for j starting at $j = x$. The computations are repeated for values of j that are increased by y until $j > z$.

The following output parameters are printed out by the computer:

V(1, J), HR(1, J)
V(N, J), HL(N, J)
V(N2, J), HR(N2, J), HL(N2, J)

For a specific example, the fluid system constants appearing in the computer program were chosen nominally to match experimental conditions in the Lewis line-dynamics facility shown schematically in figure 11(b) (p. 20). Two sets of line conditions were used. The first (case I) was chosen in conjunction with an analytical study of the effect of the size of the input amplitude on the linearity of the perturbation imposed on the mean flow and pressure in the line. The second (case II) was chosen to match conditions frequently obtained in the line.

The constants listed in table II were employed. A simplified computer flow diagram for this example is presented in figure 21. A complete listing of the Fortran IV program for this example is available[12].

TABLE II. - CONSTANTS FOR EXAMPLE

	Case I	Case II
C, ft/sec	3800	3800
L, ft	68	68
BO1 = BO	0.8	0.65
BA	0.48, 0.32, 0.20, 0.15	0.08, 0.22, 0.36, 0.50
BN	1.0	0.42
VO, ft/sec	14	6.5
HEN, ft	520	360
HEX	0	0
g, ft/sec^2	32.2	32.2
N	17	17
K	1	2
M	170	400
NN	16	32
F, cps	40	70
N2	5	5
ERROR	0.00001	0.00001

APPENDIX C

DIGITAL MODEL FOR TAPERED HYDRAULIC LINE

The system chosen for the tapered hydraulic line is shown in figure 17 (p. 26). This system consists of a variable-area orifice at A and a fixed orifice at B connected by a tapered line with a linear diameter variation between A and B. The line is bounded by pressure reservoirs. The tapered line was approximated by a line composed of a discrete number of constant-diameter sections of decreasing (or increasing) diameter.

To keep the solution general, N - 2 diameter discontinuities are inserted, which result in N discontinuities and N - 1 sections of straight conduit.

The conduit is so divided that the wave travel times between adjacent discontinuities are multiples of the working time interval, and the diameter of the approximating straight section of conduit is equal to the average diameter of the section that it is approximating.

It is assumed that the wave velocity remains constant over the entire conduit, which is equivalent to assuming that the ratio of the conduit diameter to wall thickness remains constant (a reasonable assumption from a structural viewpoint). The conduit is thus approximated by N - 1 sections of equal length. The diameter of the approximating straight section is given by

$$D(I) = DA + \left(\frac{DB - DA}{2}\right)\left(\frac{2I - 1}{N - 1}\right)$$

Initial values for velocity and pressure to the right and left of each discontinuity must be inserted into the computer, or the equations for the computation of these quantities from initial steady conditions must be integrated into the computer program.

The disturbing function can either be read into the computer as a function of time or it can be computed. For this model a sinusoidal orifice coefficient for the inlet (at A) is taken to be

$$B = BO + BA\ \mathrm{SIN}(2\pi FJ\Delta t)$$

The orifice coefficient for the outlet is constant and denoted by BN.

The following data are needed as input to the computer:

DA diameter at A, ft

DB diameter at B, ft

CA wave velocity in conduit, ft/sec

L	length of line, ft
BO1	steady-state input orifice coefficient
BO	mean orifice coefficient when subjected to sinusoidal perturbations
BA	amplitude of orifice coefficient perturbation
F	frequency of pressure perturbation
BN	output orifice coefficient
HEN	pressure of input reservoir, ft
QO	steady-state flow rate, ft
g	acceleration due to gravity, ft/sec
N	number of discontinuities
NN	number of working time increments for wave to travel length of conduit
K	number of working time increments between discontinuities
M	total number of time increments for which computations will be made
N2	number of diameter discontinuity for which pressure and velocity data will be printed

The following output parameters are printed out by the computer:

VR(1, J), HR(1, J)
VL(N, J), HL(N, J)
VL(N2, J), HL(N2, J)
HR(N2, J)

For a specific example the effect of line taper on the response of a particular system of the type shown in figure 17 (p. 26) was computed. Calculations were made for nine different degrees of line taper. The specific data used in this analysis were as follows:

CA = 3000 ft/sec
L = 50 ft
HEN = 660 ft
QO = 6.4 cu ft/sec
N = 11
NN = 10
M = 300

Table III gives the end diameters and conditions for the various tapers studied.

These cases were studied for frequencies of 5, 7.5, 10, 12.5, 15, 17.5, 20, 22.5, and 25 cps.

Figure 22 gives a simplified computer flow diagram for this example. A listing of the Fortran IV program for this example is available[12].

TABLE III. - END DIAMETERS AND CONDITIONS FOR TAPER STUDIES

DA	DB	BO	BA	BN
1	1	1	0.1	0.2
.9	1.1	1.235	.1235	.165
.8	1.2	1.562	.1562	.139
.7	1.3	2.041	.2041	.1185
.6	1.4	2.78	.278	.102
.5	1.5	4.0	.4	.089
1.1	.9	.825	.0825	.247
1.2	.8	.695	.0695	.313
1.3	.7	.593	.0593	.408

APPENDIX D

SYMBOLS

A line area, sq ft

$A(i)$ line area of i, sq ft

$A(j)$ line area of j, sq ft

A_O orifice area, sq ft

A_1 line area to left of diameter discontinuity, sq ft

A_2 line area to right of diameter discontinuity, sq ft

a, b, c polynomial coefficients

B orifice coefficient, $(ft/sec^2)^{1/2}$

B_F friction orifice coefficient, $(ft/sec^2)^{1/2}$

B_1 orifice coefficient before wave action, $(ft/sec^2)^{1/2}$

B_2 orifice coefficient after wave action, $(ft/sec^2)^{1/2}$

C sonic velocity in line, ft/sec

C_D orifice discharge coefficient

$C(i)$ wave velocity in line i, ft/sec

$C(j)$ wave velocity in line j, ft/sec

C_1 wave velocity to left of discontinuity, ft/sec

C_2 wave velocity to right of discontinuity, ft/sec

D line diameter, ft

$\bar{D}$ mean diameter of tapered line, ft

E_c elastic modulus of conveyer, lb/sq ft

E_f elastic modulus of fluid, lb/sq ft

f Darcy friction factor

f function

g acceleration due to gravity, ft/sec^2

H pressure head, ft

HEN pressure head of inlet reservoir, ft

HEX pressure head of outlet reservoir, ft

H_1 pressure head to left of discontinuity before wave action, ft

H_2 pressure head to right of discontinuity before wave action, ft

H_{11} pressure head to left of discontinuity after wave action, ft

H_{22} pressure head to right of discontinuity after wave action, ft

ΔH pressure head change across wave or orifice, ft

ΔH_1 wave impinging from left of discontinuity before wave action, ft

ΔH_2 wave impinging from right of discontinuity before wave action, ft

ΔH_{11}	wave leaving discontinuity on left side after wave action, ft
ΔH_{22}	wave leaving discontinuity on right side after wave action, ft
Δh_L	pressure head loss over small line length, ft
J	time subscript
K	number of working time increments between discontinuities
L	line length, ft
P	pressure in line, lb/sq ft
ΔP	pressure wave, lb/sq ft
Q	volume flow rate, cu ft/sec
R	area ratio at diameter discontinuity
R(i)	reflection coefficient in line i
T(i)	transmission coefficient in line i
t	time, sec
Δt	incremental time interval, sec
t^-	short time before wave action, sec
t^+	short time after wave action, sec
V	velocity in line, ft/sec
VE_1	average velocity of moving end before wave action, ft/sec
VE_2	average velocity of moving end after wave action, ft/sec
VO	velocity in line adjacent to moving orifice, relative to orifice, ft/sec
V_1	velocity in line to left of discontinuity before wave action, ft/sec
V_2	velocity in line to right of discontinuity before wave action, ft/sec
V_{11}	velocity in line to left of discontinuity after wave action, ft/sec
V_{22}	velocity in line to right of discontinuity after wave action, ft/sec
x	position, ft
δ	line taper factor (eqs. (31)), ft
ρ	mass density of fluid, slugs/ft^3
τ	wall thickness of conduit, ft

FOOTNOTES AND REFERENCES

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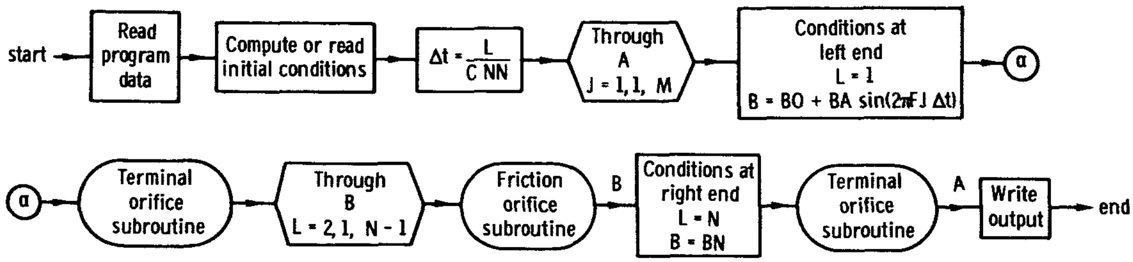

Figure 21. - Computer flow diagram for hydraulic line with oscillating input orifice.

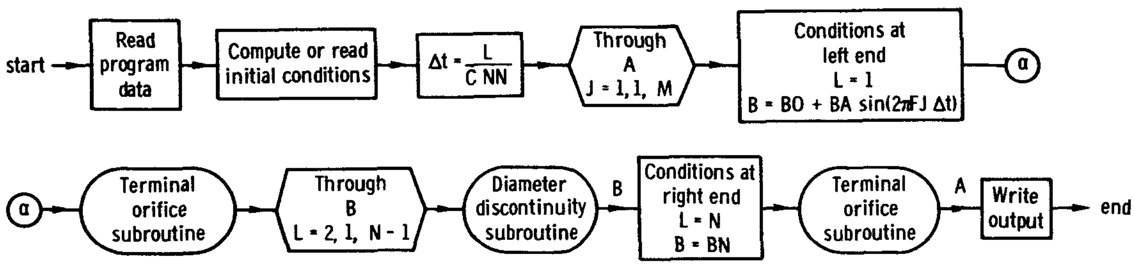

Figure 22. - Computer flow diagram for tapered hydraulic line with oscillating input orifice.